UNIT - IV

Numerical Methods - I

The expression $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ where a's are constants $(a_0 \neq 0)$ and n is an integer, is called a polynomial in x of degree n.

The polynomial $f(x) = 0$ is called an "Algebraic Equation".

eg. $x^4 - x - 9 = 0$

The polynomial $f(x) = 0$; which has trignometric, or logarithmic or exponential function involved is called as "transcendental equations."

I] Newton - Raphson Method :-

Working Rule :

① Let $f(x) = 0$ be the given equation; and get $f(x)$.

② Find $f'(x)$.

③ Find $x = a$ and $x = b$ two values of x such that $f(a)$ and $f(b)$ has opposite signs.

④ Compare $|f(a)|$ and $|f(b)|$ and if $f(a) < f(b)$, then let $x_0 = a$

⑤ Find x_1 ; by Newton - Raphson formula

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

put $n = 0, 1, 2, \cdots$ in the above formula and find $x_1, x_2 \ldots$ and so on till we get desired accuracy of root.

Ex.] Find the real root of the equation $x^4 - x - 9 = 0$, correct upto three decimal by Newton - Raphson method.

Soln:- Let $f(x) = x^4 - x - 9$ ——— (1)

$f'(x) = 4x^3 - 1$ ——— (2)

Eqn (1) $\Rightarrow$

$$f(0) = (0)^4 - 0 - 9 = -9 \quad (-ve)$$
$$f(1) = (1)^4 - 1 - 9 = -9 \quad (-ve)$$
$$f(2) = (2)^4 - 2 - 9 = 5 \quad (+ve)$$

Here, f(1) and f(2) has opposite signs.

$\Rightarrow$ Root lies between 1 and 2

Now, $|f(2)| < |f(1)|$

$\therefore \boxed{x_0 = 2}$

By Newton - Raphson formula,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad \text{———} \quad (3)$$

Now, $f(x_0) = f(2) = 5$

$f'(x_0) = f'(2) = 4(2)^3 - 1 = 31$ ----- {from (2)}

$\therefore$ put $n = 0$; in eqn (3), we get

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 2 - \frac{5}{31}$$

$$\boxed{x_1 = 1.8387}$$

~~put~~ $f(x_1) = f(1.8387) = (1.8387)^4 - 1.8387 - 9$

$= 0.5912$

$f'(x_1) = f'(1.8387) = 4(1.8387)^3 - 1$

$= 23.8652$

put $n = 1$, in eqⁿ 3, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.8387 - \frac{0.5912}{23.8652}$$

$$\boxed{x_2 = 1.8139}$$

Now, $f(x_2) = f(1.8139) = (1.8139)^4 - 1.8139 - 9 = 0.0117$

$f'(x_2) = f'(1.8139) = 4(1.8139)^3 - 1 = 22.8726$

put $n = 2$, in eqⁿ 3, we get

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.8139 - \frac{0.0117}{22.8726}$$

$$\boxed{x_3 = 1.8133}$$

Now, $f(x_3) = f(1.8133) = (1.8133)^4 - 1.8133 - 9 = -1.9818 \times 10^{-3}$

$f'(x_3) = f'(1.8133) = 4(1.8133)^3 - 1 = 22.8489$

put $n = 3$, in eqⁿ 3, we get

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 1.8133 - \frac{(-1.9818 \times 10^{-3})}{22.8489}$$

$$\boxed{x_4 = 1.8133}$$

Since, x_3 and x_4 are almost same ; we stop here to find the approximations of root.

$\therefore$ The required root for the equation $x^4 - x - 9 = 0$ is

$$\boxed{x = 1.8133}$$

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Ex.] Find the real root of the equation $xe^x - 3 = 0$, correct upto four decimal place by Newton-Raphson method.

Soln:- Given; $xe^x - 3 = 0$

$\therefore f(x) = xe^x - 3$ ——— ①

$f'(x) = xe^x + e^x(1) - 0$

$= xe^x + e^x$

$f'(x) = e^x(x+1)$ ——— ②

Eqn ① $\Rightarrow$ $f(x) = xe^x - 3$

$f(0) = 0e^0 - 3 = -3$ (-ve)

$f(1) = 1 \cdot e^1 - 3 = -0.2817$ (-ve)

$f(2) = 2e^2 - 3 = 11.7781$ (+ve)

Here, $f(1)$ and $f(2)$ has opposite sign.

$\Rightarrow$ Root lies between 1 and 2

Also, $|f(1)| < |f(2)|$

$\therefore \boxed{x_0 = 1}$

By Newton - Raphson formula,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad ——— ③$$

$f(x_0) = f(1) = -0.2817$

$f'(x_0) = f'(1) = e^1(1+1) = 5.4365$

eqn ③ $\Rightarrow$ for $n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{(-0.2817)}{5.4365}$$

$$\boxed{x_1 = 1.0518}$$

$f(x_1) = f(1.0518) = (1.0518)e^{1.0518} - 3 = 0.0110$

$f'(x_1) = f'(1.0518) = e^{1.0518}(1.0518+1) = 5.8738$

put $n = 1$; in eqn ③, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.0518 - \frac{0.0110}{5.8738}$$

$$\boxed{x_2 = 1.0499}$$

Now, $f(x_2) = f(1.0499) = (1.0499)e^{1.0499} - 3 = -5.21 \times 10^{-5}$

$f'(x_2) = f'(1.0499) = e^{1.0499}(1.0499+1) = 5.8573$

put $n=2$; in eqn ③, we get

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.0499 - \frac{(-5.21 \times 10^{-5})}{5.8573}$$

$$\boxed{x_3 = 1.0499}$$

The required root for the equation is

$$\boxed{x = 1.0499}$$

Ex.] Solve by Newton - Raphson method $e^x - 4x = 0$.

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Soln :- Given equation is $e^x - 4x = 0$

$$\Rightarrow f(x) = e^x - 4x \quad \text{--- (1)}$$

$$\therefore f'(x) = e^x - 4 \quad \text{--- (2)}$$

Eqn (1) $\Rightarrow f(x) = e^x - 4x$

$f(0) = e^0 - 4\times 0 = 1$ (+ve)

$f(1) = e^1 - 4 = -1.2817$ (−ve)

$\therefore f(0)$ and $f(1)$ has opposite signs.

$\Rightarrow$ Root lies between 0 and 1.

Comparing, $|f(0)|$ and $|f(1)|$;

$|f(0)| < |f(1)|$

$\Rightarrow \boxed{x_0 = 0}$

By Newton - Raphson formula,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad \text{--- (3)}$$

$\therefore f(x_0) = f(0) = 1$

$f'(x_0) = f'(0) = e^0 - 4 = 1 - 4 = -3$

put $n = 0$; in eqn (3), we get

$$x_1 = x_0 + \frac{f(x_0)}{f'(x_0)}$$

$$= 0 - \frac{1}{(-3)}$$

$$\boxed{x_1 = 0.3333}$$

Now, $f(x_1) = f(0.3333) = e^{0.3333} - 4(0.3333) = 0.0623$

$f'(x_1) = f'(0.3333) = e^{0.3333} - 4 = -2.6044$

put $n = 1$; in eqn (3), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.3333 - \frac{0.0623}{(-2.6044)}$$

$$\boxed{x_2 = 0.3572}$$

Now, $f(x_2) = f(0.3572) = e^{0.3572} - 4 \times 0.3572 = 5.2170 \times 10^{-4}$

$f'(x_2) = f'(0.3572) = e^{0.3572} - 4 = -2.5706$

put $n = 2$; in eqn (3), we get

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.3572 - \frac{5.2170 \times 10^{-4}}{(-2.5706)}$$

$$\boxed{x_3 = 0.3574}$$

Now; $f(x_3) = f(0.3574) = e^{0.3574} - 4 \times 0.3574 = 7.5985 \times 10^{-6}$

$f'(x_3) = f'(0.3574) = e^{0.3574} - 4 = -2.5703$

put $n = 3$; in eqn (4), we get

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.3574 - \frac{7.5985 \times 10^{-6}}{(-2.5703)}$$

$$\boxed{x_3 = 0.3574}$$

∴ The required soln is

$$\boxed{x = 0.3574}$$

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Ex.] Find the root of the equation $\cos x = 3x - 1$ by Newton-Raphson method correct upto four decimal places.
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Note: Whenever there is trignometric function in your equation; convert the calculator into radian mode from degree to perform calculations.

Soln:- Given, $\cos x = 3x - 1$

$$\Rightarrow \cos x - 3x + 1 = 0$$

$$\therefore f(x) = \cos x - 3x + 1 \quad \text{--- (1)}$$

$$f'(x) = -\sin x - 3 \quad \text{--- (2)}$$

Now, eqn (1) $\Rightarrow$ $f(x) = \cos x - 3x + 1$

$$f(0) = \cos 0 - 3(0) + 1 = 2 \ (+ve)$$

$$f(1) = \cos 1 - 3(1) + 1 = -1.4596 \ (-ve)$$

$\Rightarrow$ $f(0)$ and $f(1)$ has opposite sign.

$\therefore$ Root lies between 0 and 1.

Comparing, $|f(0)|$ and $|f(1)|$

$$|f(1)| < |f(0)|$$

$$\Rightarrow \boxed{x_0 = 1}$$

By Newton-Raphson method,

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}} \quad \text{--- (3)}$$

$f(x_0) = f(1) = -1.4596$

$f'(x_0) = f'(1) = -\sin 1 - 3 = -3.8414$

put $n = 0$; in eqn (3), we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{(-1.4596)}{(-3.8414)}$$

$$\boxed{x_1 = 0.6200}$$

Now, $f(x_1) = f(0.6200) = \cos(0.6200) - 3(0.6200) + 1$

$= -0.0461$

$f'(x_1) = f'(0.6200) = -\sin(0.6200) - 3$

$= -3.5810$

put $n = 1$; in eqn (3), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.6200 - \frac{(-0.0461)}{-3.5810}$$

$$\boxed{x_2 = 0.6071}$$

Now, $f(x_2) = f(0.6071) = \cos(0.6071) - 3(0.6071) + 1$

$= 5.8845 \times 10^{-6}$

$f'(x_2) = f'(0.6071) = -\sin(0.6071) - 3$

$= -3.5704$

put $n = 2$; in eqn (3); we get

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.6071 - \frac{5.8845 \times 10^{-6}}{-3.5704}$$

$$\boxed{x_3 = 0.6071}$$

$\therefore$ The required solution for the given equation is

$$\boxed{x = 0.6071}$$

II] REGULA-FALSI METHOD

OR METHOD OF FALSE POSITION

Working Rule:-

① Let $f(x) = 0$; be the given eqⁿ, find $f(x)$.

② Find $x = a$ and $x = b$ two values of x such that $f(a)$ and $f(b)$ has opposite signs.

③ We take $x_0 = a$ and $x_1 = b$; as the initial approximations,

④ By Regula - Falsi formula ;

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

⑤ Find $f(x_2)$

⑥ Check the sign of $f(x_2)$ with $f(x_1)$ and $f(x_0)$ and choose the opposite sign to $f(x_2)$ to go in next approximations.
(i.e. if $f(x_2)$ and $f(x_1)$ has opposite sign then solve for x_3 by taking x_2 and x_1 as initial approximations.)

⋮

Continue this process till you get the desired accuracy of root.

Ex. 1] Find the real root of the equation $3x-1=\cos x$ by Regula-Falsi method.

Note: Convert the calculator in radian mode for this problem.

Soln:- Given eqn is $3x-1=\cos x$

$\Rightarrow \quad 3x-1-\cos x=0$

or $\quad \cos x-3x+1=0$ ——— (1)

$\therefore \quad f(x)=\cos x-3x+1$ ——— (2)

$f(x_0)=f(0)=\cos 0-3\times 0+1 \quad = 2 \quad (+ve)$

$f(x_1)=f(1)=\cos 1-3(1)+1 \quad = -1.4596 \quad (-ve)$ ——— (3)

$\Rightarrow$ $f(0)$ and $f(1)$ has opposite signs.

$\therefore$ Root lies between 0 and 1.

Let $\boxed{x_0=0}$ and $\boxed{x_1=1}$

By Regula-Falsi method,

$$x_2=\frac{x_0 f(x_1)-x_1 f(x_0)}{f(x_1)-f(x_0)}$$

$$=\frac{0(-1.4596)-1(2)}{(-1.4596)-(2)}$$

$$\boxed{x_2=0.5781}$$

Now, $f(x_2)=f(0.5781)$

$=\cos(0.5781)-3(0.5781)+1$

$=0.1032 \quad (+ve)$

$\because$ $f(x_2)$ and $f(x_1)$ has opposite sign

$\therefore$ Root lies between x_2 and x_1

$$\therefore \quad x_3=\frac{x_1 f(x_2)-x_2 f(x_1)}{f(x_2)-f(x_1)}$$

$$=\frac{1(0.1032)-(0.5781)(-1.4596)}{0.1032-(-1.4596)}$$

$\boxed{x_3 = 0.6059}$

$f(x_3) = \cos(0.6059) - 3(0.6059) + 1 = 4.2898 \times 10^{-3}$ (+ve)

$\therefore$ $f(x_3)$ and $f(x_1)$ has opposite sign.

$\therefore$ Root lies between x_3 and x_1

$$\therefore x_4 = \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)}$$

$$= \frac{1 \times (4.2898 \times 10^{-3}) - 0.6059(-1.4596)}{(4.2898 \times 10^{-3}) - (-1.4596)}$$

$\boxed{x_4 = 0.6070}$

$f(x_4) = f(0.6070) = \cos(0.6070) - 3(0.6070) + 1$

$= 3.6292 \times 10^{-4}$ (+ve)

$\therefore$ $f(x_4)$ and $f(x_1)$ has opposite sign.

$$\therefore x_5 = \frac{x_1 f(x_4) - x_4 f(x_1)}{f(x_4) - f(x_1)}$$

$$= \frac{1(3.6292 \times 10^{-4}) - 0.6070(-1.4596)}{(3.6292 \times 10^{-4}) - (-1.4596)}$$

$\boxed{x_5 = 0.6070}$

$\therefore$ The required soln of the given eqn is

$\boxed{x = 0.6070}$

Ex.] Find the root of the equation $x\log_{10} x - 1.2 = 0$ by Regula-Falsi method. 5-18

Soln:- Given; $x\log_{10} x - 1.2 = 0$

$\therefore f(x) = x\log_{10} x - 1.2$ —— (1)

$f(0) = 0.\log_{10} 0 - 1.2 = -1.2$ (-ve)

$f(1) = 1.\log_{10} 1 - 1.2 = -1.2$ (-ve)

$f(x_0) = f(2) = 2\log_{10} 2 - 1.2 = -0.5979$ (-ve)

$f(x_1) = f(3) = 3\log_{10} 3 - 1.2 = 0.2313$ (+ve)

Here, $f(2)$ and $f(3)$ has opposite signs.

$\Rightarrow$ Root lies between 2 and 3.

$\therefore$ Let $\boxed{x_0 = 2}$ and $\boxed{x_1 = 3}$

By Regula-falsi method,

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{2(0.2313) - 3(-0.5979)}{(0.2313) - (-0.5979)}$$

$$\boxed{x_2 = 2.7210}$$

$f(x_2) = f(2.7210) = 2.7210 \cdot \log_{10}(2.7210) - 1.2 = -0.0171$ (-ve)

Now; $f(x_1)$ and $f(x_2)$ has opposite sign.

$\therefore$ Root lies betn x_1 and x_2

$$\therefore x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$= \frac{3(-0.0171) - 2.7210(0.2313)}{-0.0171 - 0.2313}$$

$$\boxed{x_3 = 2.7402}$$

$$f(x_3) = f(2.7402) = 2.7402 \cdot \log_{10} 2.7402 - 1.2 = -3.8904 \times 10^{-4}$$
(-ve)

Now, $f(x_1)$ and $f(x_3)$ has opposite sign.

$\therefore$ Root lies between x_1 and x_3

$$x_4 = \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)}$$

$$= \frac{3(-3.8904 \times 10^{-4}) - 2.7402(0.2313)}{(-3.8904 \times 10^{-4}) - (0.2313)}$$

$$\boxed{x_4 = 2.7406}$$

$$f(x_4) = f(2.7406) = 2.7406 \cdot \log_{10} 2.7406 - 1.2 = -4.0202 \times 10^{-5}$$
(-ve)

$\therefore$ ~~Ro~~ $f(x_1)$ and $f(x_4)$ has opposite sign.

$\therefore$ Root lies between x_1 and x_4.

$$\therefore x_5 = \frac{x_1 f(x_4) - x_4 f(x_1)}{f(x_4) - f(x_1)}$$

$$= \frac{3(-4.0202 \times 10^{-5}) - 2.7406(0.2313)}{(-4.0202 \times 10^{-5}) - (0.2313)}$$

$$\boxed{x_5 = 2.7406}$$

The required soln is $\boxed{x = 2.7406}$.

Ex.] Find the real root by method of false position of $x - \cos x = 0$ correct upto four decimal.

Solⁿ:- Given ; $x - \cos x = 0$

$\therefore f(x) = x - \cos x$ —— (1)

$f(x_0) = f(0) = 0 - \cos 0 = -1$ (−ve)

$f(x_1) = f(1) = 1 - \cos 1 = 0.4596$ (+ve)

Here, $f(0)$ and $f(1)$ has opposite sign

$\Rightarrow$ Root lies between 0 and 1.

$\therefore \boxed{x_0 = 0}$, $\boxed{x_1 = 1}$

By Regula - Falsi method,

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{0 \times 0.4596 - 1(-1)}{0.4596 - (-1)}$$

$$\boxed{x_2 = 0.6851}$$

$f(x_2) = f(0.6851) = 0.6851 - \cos(0.6851) = -0.0892$ (−ve)

Now, $f(x_1)$ and $f(x_2)$ has opposite sign

$\Rightarrow$ Root lies between x_1 and x_2

$$\therefore x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$$

$$x_3 = \frac{1 \times (-0.0892) - 0.6851(0.4596)}{(-0.0892) - (0.4596)}$$

$$\boxed{x_3 = 0.7362}$$

$f(x_3) = f(0.7362) = 0.7362 - \cos(0.7362) = -4.8255 \times 10^{-3}$ (−ve)

$\therefore f(x_1)$ and $f(x_3)$ has opposite sign.

$\therefore$ Root lies between x_1 and x_3.

$$x_4 = \frac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)}$$

$$= \frac{1 \times (-4.8255 \times 10^{-3}) - 0.7362(0.4596)}{-4.8255 \times 10^{-3} - 0.4596}$$

$$\boxed{x_4 = 0.7389}$$

$f(x_4) = f(0.7389) = 0.7389 - \cos(0.7389) = -3.0982 \times 10^{-4}$ $(-ve)$

$\therefore$ $f(x_1)$ and $f(x_4)$ has opposite sign

$\therefore$ Root lies between x_1 and x_4

$$\therefore x_5 = \frac{x_1 f(x_4) - x_4 f(x_1)}{f(x_4) - f(x_1)}$$

$$= \frac{1 \times (-3.0982 \times 10^{-4}) - 0.7389(0.4596)}{-3.0982 \times 10^{-4} - 0.4596}$$

$$\boxed{x_5 = 0.7390}$$

$f(x_5) = f(0.7390) = 0.7390 - \cos(0.7390) = -1.4247 \times 10^{-4}$ $(-ve)$

$\therefore$ $f(x_1)$ and $f(x_5)$ has opposite sign

$\therefore$ Root lies between x_1 and x_5

$$x_6 = \frac{x_1 \cdot f(x_5) - x_5 f(x_1)}{f(x_5) - f(x_1)}$$

$$= \frac{1 \times (-1.4247 \times 10^{-4}) - (0.7390)(0.4596)}{(-1.4247 \times 10^{-4}) - (0.4596)}$$

$$\boxed{x_6 = 0.7390}$$

$\therefore$ The required soln of the equation is

$$\boxed{x = 0.7390}$$

CROUT'S METHOD

Working Rule:

① Write down the given equation in matrix form,

$$AX = B \quad —①$$

② Let $A = LU \quad —②$

where,

$$L = \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ A_3 & B_3 & C_3 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & B_1 & C_1 \\ 0 & 1 & C_2 \\ 0 & 0 & 1 \end{bmatrix}$$

put the value of A, L and U in eqn ② and solve to get $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2$ and C_3 and then develop matrix L and U.

③ From ① and ②,

$$AX = B$$

$$\Rightarrow LUX = B \quad —③ \quad \{\because A = LU\}$$

Let $UX = V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \quad —④$

$\therefore$ Eqn ③ $\Rightarrow LV = B$

put the value of L, V and B and solve to develop matrix V.

④ Finally, eqn ④ $\Rightarrow UX = V$

put the value of U, X and V and solve to develop matrix X.

Ex.] Solve the equation by Crout's method

$$4x + y - z = 13$$
$$3x + 5y + 2z = 21$$
$$2x + y + 6z = 14$$

Soln:- Given equations are -

$$\left.\begin{aligned} 4x + y - z &= 13 \\ 3x + 5y + 2z &= 21 \\ 2x + y + 6z &= 14 \end{aligned}\right\} \quad \text{--- (1)}$$

Eqn (1) in matrix form $AX = B$ --- (2)

$$\begin{bmatrix} 4 & 1 & -1 \\ 3 & 5 & 2 \\ 2 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 21 \\ 14 \end{bmatrix}$$

where,

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 5 & 2 \\ 2 & 1 & 6 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 13 \\ 21 \\ 14 \end{bmatrix}$$

Now, let $A = LU$ --- (3)

where,

$$L = \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ A_3 & B_3 & C_3 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & B_1 & C_1 \\ 0 & 1 & C_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ eqn (3) $\Rightarrow$ $A = LU$

$$\begin{bmatrix} 4 & 1 & -1 \\ 3 & 5 & 2 \\ 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ A_3 & B_3 & C_3 \end{bmatrix} \begin{bmatrix} 1 & B_1 & C_1 \\ 0 & 1 & C_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & -1 \\ 3 & 5 & 2 \\ 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} A_1 & A_1B_1 & A_1C_1 \\ A_2 & A_2B_1 + B_2 & A_2C_1 + B_2C_2 \\ A_3 & A_3B_1 + B_3 & A_3C_1 + B_3C_2 + C_3 \end{bmatrix}$$

Equating the values, we get

$\boxed{A_1 = 4}$, $\boxed{A_2 = 3}$, $\boxed{A_3 = 2}$

$A_1 B_1 = 1$

$\Rightarrow \boxed{B_1 = \frac{1}{A_1} = \frac{1}{4}}$

$A_1 C_1 = -1$

$\boxed{C_1 = \frac{-1}{A_1} = \frac{-1}{4}}$

$A_2 B_1 + B_2 = 5$

$3(1/4) + B_2 = 5$

$B_2 = 5 - \frac{3}{4}$

$\boxed{B_2 = 17/4}$

$A_2 C_1 + B_2 C_2 = 2$

$3(-1/4) + \left(\frac{17}{4}\right) C_2 = 2$

$\boxed{C_2 = 11/17}$

$A_3 B_1 + B_3 = 1$

$2(1/4) + B_3 = 1$

$B_3 = 1 - 2/4$

$\boxed{B_3 = 1/2}$

$A_3 C_1 + B_3 C_2 + C_3 = 6$

$2(-1/4) + 1/2(11/17) + C_3 = 6$

$\boxed{C_3 = \frac{105}{17}}$

$$\therefore L = \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 3 & 17/4 & 0 \\ 2 & 1/2 & 105/17 \end{bmatrix} \quad \text{——} \ (4)$$

$$U = \begin{bmatrix} 1 & B_1 & C_1 \\ 0 & 1 & C_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1/4 & -1/4 \\ 0 & 1 & 11/17 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{——} \ (5)$$

From eqn (1) and (2),

$AX = B$

$\Rightarrow LUX = B$ $\quad \therefore \{A = LU\}$ —— (6)

Let $UX = V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$ —— (7)

eqn (6) $\Rightarrow$ $LV = B$

$$\begin{bmatrix} 4 & 0 & 0 \\ 3 & 17/4 & 0 \\ 2 & 1/2 & 105/17 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 13 \\ 21 \\ 14 \end{bmatrix}$$

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$\therefore\ 4v_1 = 13$

$3v_1 + \frac{17}{4} v_2 = 21$

$2v_1 + \frac{1}{2} v_2 + \frac{105}{17} v_3 = 14$

solving, we get. $\therefore \boxed{v_1 = 3/4}$, $\boxed{v_2 = \frac{45}{17}}$, $\boxed{v_3 = 1}$

$$\therefore V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 45/17 \\ 1 \end{bmatrix}$$

$\therefore$ Eqn (7) $\Rightarrow$

$UX = V$

$$\begin{bmatrix} 1 & 1/4 & -1/4 \\ 0 & 1 & 11/17 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/4 \\ 45/17 \\ 1 \end{bmatrix}$$

$x + \frac{1}{4} y + \left(-\frac{1}{4}\right) z = 3/4$

$y + \frac{11}{17} z = 45/17$

$z = 1$

On solving, we get

$\boxed{x = 3}$, $\boxed{y = 2}$, $\boxed{z = 1}$

$\therefore$ This is the required solution.

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Ex.] Solve the equation by Crout's method.

$$x_1 + 2x_2 + 3x_3 = 7$$
$$2x_1 + 7x_2 + 15x_3 = 26$$
$$3x_1 + 15x_2 + 41x_3 = 62$$

Soln:- Given equations are -

$$\left.\begin{aligned} x_1 + 2x_2 + 3x_3 &= 7 \\ 2x_1 + 7x_2 + 15x_3 &= 26 \\ 3x_1 + 15x_2 + 41x_3 &= 62 \end{aligned}\right\} \quad \text{--- (1)}$$

Eqn (1) can be written as matrix form

$$AX = B \quad \text{--- (2)}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 15 \\ 3 & 15 & 41 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 26 \\ 62 \end{bmatrix}$$

where,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 15 \\ 3 & 15 & 41 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 7 \\ 26 \\ 62 \end{bmatrix}$$

Now, let $A = LU$ --- (3)

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 15 \\ 3 & 15 & 41 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ A_3 & B_3 & C_3 \end{bmatrix} \begin{bmatrix} 1 & B_1 & C_1 \\ 0 & 1 & C_2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & 15 \\ 3 & 15 & 41 \end{bmatrix} = \begin{bmatrix} A_1 & A_1B_1 & A_1C_1 \\ A_2 & A_2B_1 + B_2 & A_2C_1 + B_2C_2 \\ A_3 & A_3B_1 + B_3 & A_3C_1 + B_3C_2 + C_3 \end{bmatrix}$$

Equating the value, we get

$\boxed{A_1 = 1}$, $\boxed{A_2 = 2}$, $\boxed{A_3 = 3}$

$A_1 B_1 = 2$

$$\boxed{B_1 = \frac{2}{A_1} = \frac{2}{1} = 2}$$

$A_1 C_1 = 3$

$$\boxed{C_1 = \frac{3}{A_1} = \frac{3}{1} = 3}$$

$A_2 B_1 + B_2 = 7$

$2(2) + B_2 = 7$

$$\boxed{B_2 = 3}$$

$A_2 C_1 + B_2 C_2 = 15$

$2(3) + (3) C_2 = 15$

$3 C_2 = 15 - 6$

$$\boxed{C_2 = 3}$$

$A_3 B_1 + B_3 = 15$

$3(2) + B_3 = 15$

$$\boxed{B_3 = 9}$$

$A_3 C_1 + B_3 C_2 + C_3 = 41$

$3(3) + (9)(3) + C_3 = 41$

$9 + 27 + C_3 = 41$

$$\boxed{C_3 = 5}$$

$$L = \begin{bmatrix} A_1 & 0 & 0 \\ A_2 & B_2 & 0 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 9 & 5 \end{bmatrix} \quad \text{--- } (4)$$

$$U = \begin{bmatrix} 1 & B_1 & C_1 \\ 0 & 1 & C_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{--- } (5)$$

From equation (1) and (2),

$AX = B$

$LUX = B$ —— (6) - - - $\{\text{from } A = LU\}$

Let, $UX = V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$ —— (7)

eqn (6) $\Rightarrow$ $LV = B$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 9 & 5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 26 \\ 62 \end{bmatrix}$$

$$V_1 = 7$$

$$2V_1 + 3V_2 = 26$$

$$3V_1 + 9V_2 + 5V_3 = 62$$

On solving above eq^n, we get.

$$V_1 = 7$$

$$V_2 = 4$$

$$V_3 = 1$$

$$\therefore \quad V = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}$$

$\therefore$ eq^n (7) $\Rightarrow$ $UX = V$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}$$

$$x_1 + 2x_2 + 3x_3 = 7$$

$$x_2 + 3x_3 = 4$$

$$x_3 = 1$$

$\therefore$ On solving, we get

$$\boxed{x_1 = 2}, \quad \boxed{x_2 = 1}, \quad \boxed{x_3 = 1}$$

This is the required solution.

GAUSS - SEIDAL METHOD

Working Rule:

1. Write down the given equation in such a way that the coefficients of x, y and z are larger in sequence.
2. Find the value of x, y, z from each of equation.
3. First take initial approximations $x_0 = 0$, $y_0 = 0$, $z_0 = 0$ and find x_1, y_1, z_1.
4. Then with the help of x_1, y_1, z_1 find x_2, y_2, z_2.
5. Continue this process of finding the roots till you get the repeated roots.

Ex.] ~~Given~~ Solve by Gauss - Seidal method

5-16

$$2x - 3y + 20z = 25$$
$$20x - y - 2z = 17$$
$$3x + 20y - z = -18$$

Soln:- Arrange the given equations as

$$20x - y - 2z = 17 \quad \text{--- (1)}$$
$$3x + 20y - z = -18 \quad \text{--- (2)}$$
$$2x - 3y + 20z = 25 \quad \text{--- (3)}$$

Eqn (1) $\Rightarrow$ $x = \frac{1}{20}[17 + y + 2z]$ --- (4)

Eqn (2) $\Rightarrow$ $y = \frac{1}{20}[-18 - 3x + z]$ --- (5)

Eqn (3) $\Rightarrow$ $z = \frac{1}{20}[25 - 2x + 3y]$ --- (6)

Let initial approximations are
$x_0 = 0$, $y_0 = 0$, $z_0 = 0$

First iteration:

Eqn (4) $\Rightarrow$ $x_1 = \frac{1}{20}[17 + y_0 + 2z_0]$

$$x_1 = \frac{1}{20}[17 + 0 + 2(0)]$$

$$\boxed{x_1 = 0.85}$$

Eqn (5) $\Rightarrow$ $y_1 = \frac{1}{20}[-18 - 3x_1 + z_0]$

$$= \frac{1}{20}[-18 - 3(0.85) + 0]$$

$$\boxed{y_1 = -1.0275}$$

Eqn ⑥ $\Rightarrow$ $z_1 = \frac{1}{20}\left[25 - 2x_1 + 3y_1\right]$

$= \frac{1}{20}\left[25 - 2(0.85) + 3(-1.0275)\right]$

$\boxed{z_1 = 1.0108}$

$\therefore$ $\boxed{x_1 = 0.85, \quad y_1 = -1.0275, \quad z_1 = 1.0108}$

Second iteration :-

Eqn ④ $\Rightarrow$

$x_2 = \frac{1}{20}\left[17 + y_1 + 2z_1\right]$

$= \frac{1}{20}\left[17 - 1.0275 + 2(1.0108)\right]$

$\boxed{x_2 = 0.8997}$

Eqn ⑤ $\Rightarrow$

$y_2 = \frac{1}{20}\left[-18 - 3x_2 + z_1\right]$

$= \frac{1}{20}\left[-18 - 3(0.8997) + 1.0108\right]$

$\boxed{y_2 = -0.9844}$

Eqn ⑥ $\Rightarrow$

$z_2 = \frac{1}{20}\left[25 - 2x_2 + 3y_2\right]$

$= \frac{1}{20}\left[25 - 2(0.8997) + 3(-0.9844)\right]$

$\boxed{z_2 = 1.0123}$

$\therefore$ $\boxed{x_2 = 0.8997, \quad y_2 = -0.9844, \quad z_2 = 1.0123}$

Third iteration :-

Eqⁿ (4) $\Rightarrow x_3 = \frac{1}{20}\left[17 + y_2 + 2z_2\right]$

$= \frac{1}{20}\left[17 + (-0.9844) + 2(-1.0123)\right]$

$\boxed{x_3 = 0.9020}$

Eqⁿ (5) $\Rightarrow y_3 = \frac{1}{20}\left[-18 - 3x_3 + z_2\right]$

$= \frac{1}{20}\left[-18 - 3(0.9020) + (-1.0123)\right]$

$\boxed{y_3 = -0.9846}$

Eqⁿ (6) $\Rightarrow z_3 = \frac{1}{20}\left[+25 - 2x_3 + 3y_3\right]$

$= \frac{1}{20}\left[+25 - 2(0.9020) + 3(-0.9846)\right]$

$\boxed{z_3 = 1.012}$

∴ The required solution is

$x = 0.9020$

$y = -0.9846$

$z = 1.012$

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Ex.] Solve by Gauss - Seidal Method S-18

$$10x + y + 2z = 13$$
$$3x + 10y + z = 14$$
$$2x + 3y + 10z = 15$$

Soln :- Arranging the given equations; we get

$$10x + y + 2z = 13 \quad \text{——— (1)}$$
$$3x + 10y + z = 14 \quad \text{——— (2)}$$
$$2x + 3y + 10z = 15 \quad \text{——— (3)}$$

Equation (1) $\Rightarrow$

$$x = \frac{1}{10}[13 - 2z - y] \quad \text{——— (4)}$$

Eqn (2) $\Rightarrow$ $y = \frac{1}{10}[14 - 3x - z]$ ——— (5)

Eqn (3) $\Rightarrow$ $z = \frac{1}{10}[15 - 2x - 3y]$ ——— (6)

Let, initial approximations are

$x_0 = 0, \quad y_0 = 0, \quad z_0 = 0$

$\therefore$ Eqn (4) $\Rightarrow$ $x_2 = \frac{1}{10}[13 - 2z_0 - y_0]$

$$= \frac{1}{10}[13 - 2(0) - (0)]$$

$$\boxed{x_1 = 1.3}$$

Eqn (5) $\Rightarrow$ $y_2 = \frac{1}{10}[14 - 3x_1 - z_0]$

$$= \frac{1}{10}[14 - 3(1.3) - 0]$$

$$\boxed{y_2 = 1.01}$$

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Eqn (6) $\Rightarrow$ $z_1 = \frac{1}{15}[15 - 2x_1 - 3y_1]$

$= \frac{1}{15}[15 - 2(1.3) - 3(1.01)]$

$\boxed{z_1 = 0.937}$

$\therefore$ $\boxed{x_1 = 1.3, \quad y_1 = 1.01, \quad z_1 = 0.937}$

Next iteration :-

Eqn (4) $\Rightarrow$ $x_2 = \frac{1}{10}[13 - y_1 - 2z_1]$

$= \frac{1}{10}[13 - 1.01 - 2(0.937)]$

$\boxed{x_2 = 1.0116}$

Eqn (5) $\Rightarrow$ $y_2 = \frac{1}{10}[14 - 3x_2 - z_1]$

$= \frac{1}{10}[14 - 3(1.0116) - 0.937]$

$\boxed{y_2 = 1.0028}$

Eqn (6) $\Rightarrow$ $\cancel{z_2 = \frac{1}{10}[15 - 2(1.0116)} -$

$z_2 = \frac{1}{10}[15 - 2(x_2) - 3y_2]$

$= \frac{1}{10}[15 - 2(1.0116) - 3(1.0028)]$

$\boxed{z_2 = 0.9969}$

$\therefore$ $\boxed{x_2 = 1.0116, \quad y_2 = 1.0028, \quad z_2 = 0.9969}$

· Next iteration :-

Eqⁿ (4) $\Rightarrow$ $x_3 = \frac{1}{10}[13 - y_2 - 2z_2]$

$= \frac{1}{10}[13 - 1.0028 - 2(0.9969)]$

$\boxed{x_3 = 1.0003}$

Eqⁿ (5) $\Rightarrow$ $y_3 = \frac{1}{10}[14 - 3x_3 - z_2]$

$= \frac{1}{10}[14 - 3(1.0003) - 0.9969]$

$\boxed{y_3 = 1.0003}$

Eqⁿ (6) $\Rightarrow$ $z_3 = \frac{1}{10}[15 - 2x_3 - 3y_3]$

$= \frac{1}{10}[15 - 2(1.0003) - 3(1.0003)]$

$\boxed{z_3 = 0.9998}$

$\therefore$ The required solution is

$x = 1.0003 \approx 1$

$y = 1.0003 \approx 1$

$z = 0.9998 \approx 1$